Math 3335 Homework Solutions

p 212 #6 $\quad F = (6x - 2e^{2x}y^2)i - 2ye^{2x}j + \cos(z)k$

a. Is F conservative?

Solution F is continuously differentiable on $\mathbb{R}^3$, and

(8)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 6x - 2e^{2x}y^2 & -2ye^{2x} & \cos z \end{vmatrix} = k(-4ye^{2x} + 4ye^{2x}) = 0$$

So F is conservative on $\mathbb{R}^3$ by Theorem 4.3.

(8)

In fact $F = \nabla\phi$, where $\frac{\partial\phi}{\partial x} = (6x - 2e^{2x}y^2)$

$$\phi(x,y,z) = 3x^2 - e^{2x}y^2 + h(y,z)$$

$$\frac{\partial\phi}{\partial y} = -2e^{2x}y + \frac{\partial h}{\partial y} = -2ye^{2x} \quad \text{so } \frac{\partial h}{\partial y} = 0, \quad h(y,z) = g(z)$$

$$\frac{\partial\phi}{\partial z} = g'(z) = \cos(z)$$

$$g(z) = \int \cos z\,dz = \sin z + c$$

$\phi(x,y,z) = 3x^2 - e^{2x}y^2 + \sin z$ is a potential for F.

b. Evaluate $\int_C F \cdot dR$ along the path C parameterized by

$$R(t) = ti + (t-1)(t-2)j + \frac{\pi}{2}t^3k, \quad 0 \le t \le 1$$

(8)

Solution Evaluation of $\int_C F \cdot dR$ from the definition is a mess.

However by Theorem 4.1

$$\int_C F \cdot dR = \phi(R(1)) - \phi(R(0))$$

$$= \phi(1, 0, \pi/2) - \phi(0, 2, 0)$$

$$= 3 + e^2 0 + 1 - (0 - 4 + 0) = 8$$

b. $R(t) = -\frac{1}{2}(t-1)i + t(3-t)j + \frac{\pi}{4}(t-1)k, \quad 1 \le t \le 3$

(8) $\quad \int_C F \cdot dR = \phi(R(3)) - \phi(R(1)) = \phi(1, 0, \pi/2) - \phi(0, 2, 0) = 8$

Math 3335 Homework Solutions

p 212 #9 a. Find a potential ϕ for

$$F = (2xyz + z^2 - 2y^2 + 1)i + (x^2z - 4xy)j + (x^2y + 2xz - 2)k$$

Solution (check that $\nabla \times F = 0$).

$$\frac{\partial \phi}{\partial x} = (2xyz + z^2 - 2y^2 + 1), \text{ so } \phi(x,y,z) = \int 2xyz + z^2 - 2y^2 + 1 \, dx$$

$$\phi(x,y,z) = x^2yz + xz^2 - 2xy^2 + x + h(y,z)$$

Then $\frac{\partial \phi}{\partial y} = x^2z - 4xy + \frac{\partial h}{\partial y} = x^2z - 4xy$ so $\frac{\partial h}{\partial y} = 0$, $h(y,z) = g(z)$

(10)

Then $\frac{\partial \phi}{\partial z} = x^2y + 2xz + g'(z) = x^2y + 2xz - 2$

so $g'(z) = -2$, $g(z) = -2z + C$

So $\phi(x,y,z) = x^2yz + xz^2 - 2xy^2 + x - 2z \; (+C)$.

b. The field $G = \frac{x}{(x^2+z^2)^2} i + \frac{z}{(x^2+z^2)^2} k$

Satisfies $\nabla \times G = 0$ except on the y-axis. Is G conservative?

Solution $G(x,y) = \nabla\left(\frac{-1}{2(x^2+z^2)}\right)$ so G is conservative.

(8)

Math 3335 Homework Solutions

p 236 #2 Find $d\vec{S}$ and dS in terms of $d\phi$ and $d\theta$, for the surface with parametric equations

$$x = (1+\cos\theta)\cos\phi$$

$$y = (1+\cos\theta)\sin\phi \qquad (x,y,z) = R(\theta,\phi)$$

$$z = \sin\theta.$$

Solution $\quad d\vec{S} = \left(\frac{\partial R}{\partial\theta}\times\frac{\partial R}{\partial\phi}\right)d\theta\, d\phi \quad dS = \left|\frac{\partial R}{\partial\theta}\times\frac{\partial R}{\partial\phi}\right| d\theta\, d\phi$

(12)

$$\frac{\partial R}{\partial\theta} = -\sin\theta\cos\phi\, i - \sin\theta\sin\phi\, j + \cos\theta\, k$$

$$\frac{\partial R}{\partial\phi} = -(1+\cos\theta)\sin\phi\, i + (1+\cos\theta)\cos\phi\, j$$

$$\frac{\partial R}{\partial\theta}\times\frac{\partial R}{\partial\phi} = \begin{vmatrix} i & j & k \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \\ -(1+\cos\theta)\sin\phi & (1+\cos\theta)\cos\phi & 0 \end{vmatrix}$$

$$= i\left(-(\cos\theta+\cos^2\theta)\cos\phi\right) - j\left((1+\cos\theta)\cos\theta\sin\phi\right) + k\left(-\sin\theta(1+\cos\theta)(\cos^2\phi+\sin^2\phi)\right)$$

$$d\vec{S} = \left(-(\cos\theta+\cos^2\theta)(\cos\phi\, i + \sin\phi\, j) - \sin\theta(1+\cos\theta)k\right)d\theta\, d\phi$$

$$dS = \sqrt{(\cos^2\theta+\sin^2\theta)(1+\cos\theta)^2}\; d\theta\, d\phi$$

$$= (1+\cos\theta)\, d\theta\, d\phi$$

Math 3335 Homework Solutions

p. 236 #10 Derive the identity $dS = (EG - F^2)^{1/2}\,du\,dv$

where $E = \left|\frac{\partial R}{\partial u}\right|^2$, $F = \frac{\partial R}{\partial u} \cdot \frac{\partial R}{\partial v}$ $G = \left|\frac{\partial R}{\partial v}\right|^2$

Solution $dS = \left|\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\right| du\,dv$

$\left|\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\right| = \left|\frac{\partial R}{\partial u}\right|\left|\frac{\partial R}{\partial v}\right| \sin\theta$ θ = angle between $\frac{\partial R}{\partial u}$ + $\frac{\partial R}{\partial v}$

10

$= \left|\frac{\partial R}{\partial u}\right|\left|\frac{\partial R}{\partial v}\right| \sqrt{1 - \cos^2\theta}$

$= \sqrt{EG - \left(\left|\frac{\partial R}{\partial u}\right|\left|\frac{\partial R}{\partial v}\right| \cos\theta\right)^2} = \sqrt{EG - \left(\frac{\partial R}{\partial u} \cdot \frac{\partial R}{\partial v}\right)^2}$

$= \sqrt{EG - F^2}$

Math 3335 Homework Solutions

p 246 # 2a. Compute $\iint_S F \cdot dS$, S = surface of cube $[-1,1]^3$

$F = xi$

Solution F is $\perp n$ except on $x = \pm 1$.

(6) $\iint F \cdot dS = \int_{-1}^{1}\int_{-1}^{1} (i)\cdot(i)\,dy\,dz + \int_{-1}^{1}\int_{-1}^{1} (-i)\cdot(-i)\,dy\,dz$

$= 8$

#14 $F = yi + (x+2)j + x^3 \sin(yz) k$, S = portion of cylinder $x^2+y^2=1$ in 1^{st} octant, below $z=1$.

$\int_S F \cdot dS = ?$

(10) Solution: Parameterize S as $R(\theta, z) = \cos\theta i + \sin\theta j + z k$.

$0 \le \theta \le \pi/2$, $0 \le z \le 1$

$\frac{\partial R}{\partial \theta} = -\sin\theta i + \cos\theta j \qquad \frac{\partial R}{\partial z} = k$

$\frac{\partial R}{\partial \theta} \times \frac{\partial R}{\partial z} = \sin\theta j + \cos\theta i$

$F(R(\theta)) \cdot \left(\frac{\partial R}{\partial \theta} \times \frac{\partial R}{\partial z}\right) = \cos\theta(\sin\theta) + \sin\theta(2 + \cos\theta)$

$= 2\sin\theta\cos\theta + 2\sin\theta$

$\iint_S F \cdot dS = \int_0^{\pi/2}\int_0^1 2\sin\theta\cos\theta + 2\sin\theta\,dz\,d\theta$

$= (\sin\theta)^2 - 2\cos\theta \Big|_0^{\pi/2} = 1 + 2 = 3$

Math 3335 Homework Solutions

Top view

p 246 #16 Torus 

a. Derive parameterization

(14). First parameterize a circle C of radius A in the xy plane, centered at O:

$$x = A\cos(u), \quad y = A\sin(u), \quad 0 \le u \le 2\pi$$

Then, for each u, parameterize a circle in the plane $y/x = \tan u$, or $x/y = \cot u$, centered at $(A\cos u, A\sin u)$ + radius a:

$$r = \sqrt{(x - A\cos u)^2 + (y - A\sin u)^2} = a\cos v,$$

$$z = a\sin v$$

Then $x - A\cos u = r\cos u = a\cos u\cos v$

$y - A\sin u = r\sin u = a\sin u\cos v$

$$x = A\cos u + a\cos u\cos v \qquad z = a\sin v \quad 0 \le v \le 2\pi$$

$$y = A\sin u + a\sin u\cos v$$

b. Area of torus $= 4\pi^2 Aa$.

(12)

$$\frac{\partial R}{\partial u} = (-A\sin u - a\sin u\cos v)i + (A\cos u + a\cos u\cos v)j$$

$$\frac{\partial R}{\partial v} = (-a\cos u\sin v)i - a\sin u\sin v\, j + a\cos v\, k$$

$$\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v} = \begin{vmatrix} i & j & k \\ -\sin u(A + a\cos v) & \cos u(A + a\cos v) & 0 \\ -a\cos u\sin v & -a\sin u\sin v & a\cos v \end{vmatrix}$$

$$= i\, a\cos v\cos u(A + a\cos v) + j a\sin u\cos v(A + a\cos v) + k\, a(\sin^2 u + \cos^2 u)\sin v(A + a\cos v)$$

$$= a(A + a\cos v)(i\cos v\cos u + j\sin u\cos v + k\sin v)$$

and $\left|\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\right| = a(A + a\cos v)$

The area of torus $= \int_0^{2\pi}\int_0^{2\pi} aA + a^2\cos v\, dv\, du = 4\pi^2 aA$